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Time-Varying Koopman Operator Theory for Nonlinear Systems Prediction

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Abstract— **This paper introduces the concept of time-varying Koopman operator to predict the flow of a nonlinear dynamical system. The Koopman operator provides a linear prediction model for nonlinear systems in a lifted space of infinite dimension. An extension of time-invariant subspace realization methods known as the time-varying Eigensystem Realization Algorithm (TVERA) in conjunction with the time-varying Observer Kalman Identification Algorithm (TVOKID) are used to derive a finite dimensional approximation of the infinite dimensional Koopman operator at each time step. An isomorphic coordinate transformations is defined to convert different system realizations from different sets of experiments into a common frame for state propagation and to extract dynamical features in the lifted space defined by the eigenvalues of the Koopman operator. Two benchmark numerical examples are considered to demonstrate the capability of the proposed approach.**

I. INTRODUCTION

Recent advances in nonlinear system identification have used the Koopman operator theoretic approach to obtain precise predictions of a nonlinear dynamical system as the output of a truncated linear dynamical system. The main idea behind the Koopman theory [1], [2] is to lift the nonlinear dynamics into a higher dimensional space where the evolution of the flow of the system can be linear. Even though the core challenge of the Koopman operator theoretic approach is to specify (directly or indirectly through decompositions) the Hilbert space of measurement functions of the state of the system, the theory has been applied for uncontrolled [3], [4] and controlled systems [5], [6] with promising results using popular subspace realization methods such as dynamic mode decomposition (DMD) and its extensions [7]. The resulting linear operator is a local approximator of the nonlinear dynamical system valid in the neighborhood of a nominal point and the domain of validity of this local linear approximation improves as the dimension of the lifting space is increased. However, one may need a very large dimensional lifting space to accurately capture the flow of the underlying nonlinear system.

One of the alternative to improve the validity region of the Koopman operator and curtail the dimension of the lifting space is to consider the linearization of the nonlinear flow about a nominal trajectory of the nonlinear system rather than a nominal point. The linearization about a nominal trajectory leads to a linear time varying (LTV) system as opposed to a linear time invariant (LTI) system for the conventional Koopman operator. However, LTV systems exhibit distinct properties, as compared to the shift invariance exhibited by LTI systems. All the subspace methods for LTI system identification exploit the fact that an infinity of system realizations exist and actually share the same Markov parameters (also known as system impulse response functions) and the eigenvalues of the state transition matrix. However, no such property exists for the LTV system. The lack of similarity transformations handicap the application of conventional subspace methods such as DMD to identify LTV systems. The literature in linear time varying system identification [8]–[15] is limited as compared to LTI system identification by the fact that there are no approaches to find similarity transformations between the model sequences. In our earlier work [13], [14], it is shown that there exist special reference frames, in which the identified models are similar to the true model, i.e., state transition matrices share the same eigenvalues. Using this key result the realizations can be compared across different data sets. This forms the basis for spectral characterization of the time varying systems and the resulting algorithm is known as the time varying eigensystem realization algorithm (TVERA).

This paper exploits the TVERA formulation in conjunction with the idea of lifted space of measurement to develop a time-varying Koopman operator. The important notion of kinematically similar (topologically equivalent) realizations is discussed and used as a tool to compare the resulting identified model. Furthermore, this paper also briefly introduces the connection between the Koopman operator and higher-order state transition matrices. It is shown that the elements of the Koopman operator corresponds to higher-order state transition matrices when a polynomial basis is used for the lifting space. This is equivalent to doing a Taylor series expansion of the nonlinear flow about a nominal trajectory with the order of expansion being equal to the order of lifting functions.

The structure of the paper is as follows: the next section introduces the time-varying version of the Koopman operator and discusses the inherent difficulties associated with the identification of time-varying Koopman operator. Section III describes the methodology to build a time-varying Koopman operator while using the TVERA algorithm and coordinate frame corrections to obtain similar system realizations for different data sets. Finally, Section IV considers two numer-

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ical examples of increasing complexity and showcases the capability of TVERA combined with the Koopman lifted space of measurements to accurately predict the output of nonlinear dynamical systems.

II. Problem Statement

Let's consider a dynamical system in a state space form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state of the system (also usually the *unknown* minimal set of variables needed to describe the evolution of the system), $\boldsymbol{u} \in \mathbb{R}^r$ is a control input to the system and $\boldsymbol{f}$ is a function of vector field that describes how the system changes at a given state in time. Let $\boldsymbol{F}$ be the flow of the dynamical system that maps the state from one time to the other:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}(\boldsymbol{x}_k, \boldsymbol{u}_k) \tag{2}$$

with $\boldsymbol{x}_k = \boldsymbol{x}(k\Delta t)$ and $\boldsymbol{u}_k = \boldsymbol{u}(k\Delta t)$. If $\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{u})$ represents a set of measurement of the pair $(\boldsymbol{x}, \boldsymbol{u})$ in the Hilbert space $\mathcal{F}$, then the infinite-dimensional Koopman operator $\mathcal{K}$ provides a linear operator for the transition of these measurements forward in time, i.e.,

$$\boldsymbol{\chi}_{k+1} = \mathcal{K}\boldsymbol{\chi}_k, \tag{3}$$

where

$$\boldsymbol{\chi}_k = \begin{bmatrix} \chi_k^1 \\ \chi_k^2 \\ \vdots \end{bmatrix} = \boldsymbol{\chi}(\boldsymbol{x}_k, \boldsymbol{u}_k) \tag{4}$$

Each $\chi_k^i = \chi^i(\boldsymbol{x}_k, \boldsymbol{u}_k)$, $i = 1, 2, \ldots$, is assumed to be observable in $\mathcal{F}$. For this purpose, the states of nonlinear system $\boldsymbol{x}_k$ are included as first n components of $\boldsymbol{\chi}_k$ [7]. Note that Eq. (3) provides an infinite dimensional LTI system version of the nonlinear flow of Eq. (2) in the measurement space $\mathcal{F}$. Since $\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}(\boldsymbol{x}_{k+1}, \boldsymbol{u}_{k+1}) = \boldsymbol{\chi}(\boldsymbol{F}(\boldsymbol{x}_k, \boldsymbol{u}_k))$, one can write

$$\mathcal{K}\boldsymbol{\chi}_k = \boldsymbol{\chi}_k \circ \boldsymbol{F}. \tag{5}$$

The central observation is to notice that there has been a trading between nonlinear dynamics in a finite-dimensional space and linear dynamics in a potentially infinite-dimensional lifted space. The Koopman operator is shown to fully capture all properties of the underlying nonlinear dynamical system provided that the state vector $\boldsymbol{x}_k$ is observable from lifted space measurements, $\boldsymbol{\chi}_k$ [3], [4].

Though the Koopman operator is ideally an infinite dimension, the measurement vector $\boldsymbol{\chi}_k$ is generally truncated to finite dimension $N >> n$. In order to obtain a linear prediction model, the following structure for $\boldsymbol{\chi}_k$ is assumed [5] which is equivalent to assuming $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u})$ to be control affine:

$$\boldsymbol{\chi}_k(\boldsymbol{x}_k, \boldsymbol{u}_k) = \begin{bmatrix} \boldsymbol{\psi}_k(\boldsymbol{x}_k) \\ \boldsymbol{u}_k \end{bmatrix}. \tag{6}$$

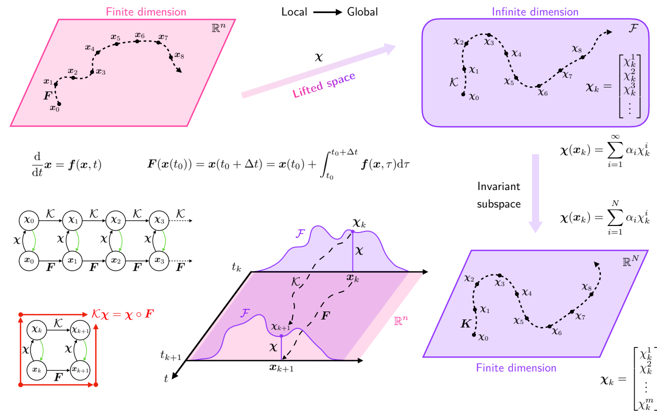

Fig. 1: Global representation of the Koopman operator

where $\boldsymbol{\psi}(\boldsymbol{x}_k)$ represents the mapping of the dynamical system states in the lifted space. Now, the problem of finding the Koopman operator can be restated as the identification of following LTI system given the time histories of $\boldsymbol{\chi}_k, \boldsymbol{u}_k$:

$$\boldsymbol{z}_{k+1} = A\boldsymbol{z}_k + B\boldsymbol{u}_k \tag{7}$$

$$\hat{\boldsymbol{\psi}}_k = C\boldsymbol{z}_k + D\boldsymbol{u}_k \tag{8}$$

Generally, $\boldsymbol{z}_k$ is assumed to be N-dimensional hidden state vector corresponding to Koopman dynamics, however, one can also estimate the dimension of $\boldsymbol{z}_k$ as part of the identification process. $\hat{\boldsymbol{\psi}}_k$ is the estimated measurement vector. The estimate for the state of the nonlinear system, denoted by $\hat{\boldsymbol{x}}_k$ can be extracted from the first n components of $\hat{\boldsymbol{\psi}}_k$. The unknown system matrices (A, B, C, D) are found such that the norm of the measurement error i.e., $\|\boldsymbol{\psi}_k - \hat{\boldsymbol{\psi}}_k\|$ is minimized. Conventional subspace decomposition methods such as the Eigensystem Realization Algorithm (ERA) [16]–[18] or Dynamic Mode Decomposition (DMD) [19]–[21] are used to provide *observable and controllable* realizations for the system matrices (A, B, C), hence performing a linearization about a single point of the dynamical system in the lifted space. In earlier work [3]–[7], it is shown that the state prediction error improves as the dimension of the lifted space, N is increased. Generally, N needs to be much larger than state dimension n for the Koopman operator to provide a good prediction of the system states.

In this work, we seek a time varying Koopman operator as an alternate mean to increase the prediction accuracy for a fixed dimension of the lifted space, i.e.,

$$\boldsymbol{\chi}_{k+1} = \mathcal{K}_k\boldsymbol{\chi}_k, \tag{9}$$

The identification of time varying Koopman operator corresponds to finding a LTV system in the lifted space, i.e.,

$$\boldsymbol{z}_{k+1} = A_k\boldsymbol{z}_k + B_k\boldsymbol{u}_k \tag{10}$$

$$\hat{\boldsymbol{\psi}}_k = C_k\boldsymbol{z}_k + D_k\boldsymbol{u}_k \tag{11}$$

Development of methods for time-varying systems have involved recursive and fast implementations of time invariant methods by exploring structural properties of the input–output realizations [22] or by generalizing several concepts in classical linear time invariant system theory consistently [8], [9]. More recent efforts [15] have concentrated on extending LTI subspace realizations methods by considering moving time windows and weighting factors on the data sequence or introducing explicit parameters to take into account the time-varying amplitude of the corresponding modes during the decomposition phase of the algorithm [23]. However, these efforts suffer from the lack of a method to find similarity transformations between the model sequences for LTV systems obtained from different experimental data sets. For example, the algorithm outlined in Ref [15] is applicable to identify a LTV system from initial condition response data. Mixed experiments, including initial condition and controlled input response experiments, result in identification of different realization, of system matrices at each time. If there were different coordinate systems defined by the Lyapunov transformation $\boldsymbol{w}_k = T_k \boldsymbol{z}_k$ whose state space realization is given by $\boldsymbol{w}_{k+1} = F_k \boldsymbol{w}_k + G_k \boldsymbol{u}_k$, along with $\hat{\boldsymbol{\psi}}_k = H_k \boldsymbol{w}_k + D_k \boldsymbol{u}_k$, then the realizations A_k, F_k are NOT similar. This is in sharp contrast to the LTI theory, where a variety of realizations (all of infinity of them, that share the same Markov parameters) share the same spectrum. Hence, the lack of a method to find a common reference frame in which different realizations for the LTV system are similar is considered the main drawback of many LTV system identification methods. In Refs [13], [14], it is shown that there exists special reference frames in which the models are similar, i.e., $\tilde{A}_k, \tilde{F}_k$ share the same eigenvalues. This special reference frame can be determined from controllability and observability matrices corresponding to different realizations of the system matrices. The resulting algorithm is known as TVERA and the next section discusses its application to obtain the time-varying Koopman operator.

III. Time-varying Koopman Operator

This section discusses the TVERA approach to obtain the time-varying Koopman operator approximation for the nonlinear system response from the time history of control inputs and measurements in lifted-space obtained from repeated experiments. The idea of repeated experiments have been introduced in Ref. [10], [24] and presented as practical methods to realize conceptual time-varying state space model identification strategies. From a perspective of generalizing the LTI subspace methods to the case of time-varying systems, a time-varying version of ERA has been developed in [13]. Additionally, it has been showed that the generalization thus made enables the identification of time-varying plant models that are in arbitrary coordinate systems at each time step and a time-varying transformation is derived to convert system states at different time into one common frame. Furthermore, an asymptotically stable observer (to remedy the problem of unbounded growth in the number of experiments), a companion algorithm, the time-varying observer/Kalman-filter system identification (TVOKID), has been developed to work alongside with TVERA for the identification of time-varying Markov parameters from experimental data [14]. This section summarizes the key ideas of the TVERA algorithm and one should refer to [13] for more details on TVERA.

A. Time-Varying Eigensystem Realization Algorithm

To get insight into the TVERA process, let us consider the solution of the difference equation of (10)

$$\boldsymbol{\psi}_k = C_k \phi_{k,0} \boldsymbol{x}_0 + \sum_{i=0}^{k-1} h(k,i) \boldsymbol{u}_i + D_k \boldsymbol{u}_k \tag{12}$$

where $\Phi(k, i+1)$ is the state-transition matrix defined as

$$\Phi(k, k_0) = \begin{cases} A_{k-1} A_{k-2} \dots A_{k_0} & \text{for } k > k_0, \\ I & \text{for } k = k_0, \\ \text{undefined} & \text{for } k < k_0. \end{cases} \tag{13}$$

and $h_{k,i}$ are the generalized Markov parameters (or pulse response matrix) defined as

$$h_{k,i} = \begin{cases} C_k \Phi(k, i+1) B_i & \text{for } i < k-1, \\ C_k B_{k-1} & \text{for } i = k-1, \\ D_k & \text{for } i = k, \\ 0 & \text{for } i > k. \end{cases} \tag{14}$$

The identification of time-varying system matrices involves the construction of a Hankel matrix $\boldsymbol{H}_k^{(p,q)}$ at each time step consisting of generalized Markov parameters,

$$\begin{aligned} \boldsymbol{H}_k^{(p,q)} &= \begin{bmatrix} h_{k,k-1} & h_{k,k-2} & \cdots & h_{k,k-q} \\ h_{k+1,k-1} & h_{k+1,k-2} & \cdots & h_{k+1,k-q} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k+p-1,k-1} & h_{k+p-1,k-2} & \cdots & h_{k+p-1,k-q} \end{bmatrix} \\ &= \boldsymbol{O}_k^{(p)} \boldsymbol{R}_{k-1}^{(q)} \end{aligned} \tag{15}$$

where $\boldsymbol{O}_k^{(p)}$ and $\boldsymbol{R}_{k-1}^{(q)}$ are the observability and controllability matrices. The generalized Markov parameters are identified through a least squares process from $\boldsymbol{\chi}_k$ time histories obtained from forced response experiments while initial condition response is used for the construction of the Hankel matrix for the first few time steps as explained in [13]. Notice that the rank of the Hankel matrix will be N for fully controllable and observable lifted space otherwise the rank of the Hankel matrix will be equal to the rank of completely observable and controllable subspace. The singular-value decomposition (SVD) of $\boldsymbol{H}_k$ allows for the identification of the current observability and controllability matrices,

$$\boldsymbol{H}_k = \boldsymbol{U}_k \boldsymbol{\Sigma}_k \boldsymbol{V}_k^{\intercal} \simeq \boldsymbol{U}_k^{(N)} \boldsymbol{\Sigma}_k^{(N)} {\boldsymbol{V}_k^{(N)}}^{\intercal} = \boldsymbol{O}_k^{(p)} \boldsymbol{R}_{k-1}^{(q)} \tag{16}$$

$$\Rightarrow \begin{cases} \boldsymbol{O}_k^{(p)} & = \boldsymbol{U}_k^{(N)} {\boldsymbol{\Sigma}_k^{(N)}}^{1/2} \\ \boldsymbol{R}_{k-1}^{(q)} & = {\boldsymbol{\Sigma}_k^{(N)}}^{1/2} {\boldsymbol{V}_k^{(N)}}^{\intercal} \end{cases} \tag{17}$$

where $\boldsymbol{\Sigma}_k^{(N)}$ contains N non-zero singular values of the Hankel matrix and matrices $\boldsymbol{U}_k^{(N)}$ and $\boldsymbol{V}_k^{(N)}$ are constructed from first N columns of $\boldsymbol{U}_k$ and $\boldsymbol{V}_k$. Finally, the following expression for the identified system matrices are obtained by considering block shifted Hankel matrices [13]

$$\hat{A}_k = {\boldsymbol{O}_{k+1}^{(p)}}^{\dagger} \boldsymbol{O}_k^{(p)\uparrow}, \; \hat{B}_k = \boldsymbol{R}_k^{(q)}[:, 1:r], \tag{18}$$

$$\hat{C}_k = \boldsymbol{O}_k^{(p)}[1:N, :], \; \hat{D}_k = h_{k,k}. \tag{19}$$

Note that $\boldsymbol{O}_k^{(p)\uparrow} = \boldsymbol{O}_{k+1}^{(p)} A_k$ is the block shifted controllability matrix.

This methodology based on TVERA allows one to construct a time-varying version of the Koopman operator for identifying the system matrices of a controlled time-varying dynamical systems. It is shown in [13] that the identified time-varying plant models that are in arbitrary coordinate systems at each time step are compatible with one another, owing the fact that they belong to the same set of experiments. If two realizations are derived from two different sets of experiments, they will not be compatible for state propagation: the state given in a certain coordinate system cannot be propagated to the next time step unless the state transition and control influence matrices are expressed in the same (or compatible) coordinate system as the initial state of interest. Moreover, two system matrices A_k and $\hat{A}_k$ are NOT similar because the system evolution takes place in two different coordinate systems. The next section summarizes an approach to find the isomorphic transformation between successive frames.

B. Frame Correction

As discussed in the previous sections, the identified $\hat{A}_k$ at each time k is not represented in the same coordinate system as the true system representation A_k and the state propagation for linear time-varying systems takes place between time-varying coordinate systems. While the system matrices do not need any type of correction during the propagation itself, two equivalent realizations A_k and $\hat{A}_k$ are not similar; rather they are topologically equivalent. Topological equivalence (or kinematic equivalence) means that there exists a sequence of invertible, square matrices T_k such that

$$\hat{A}_k = T_{k+1}^{-1} A_k T_k. \tag{20}$$

Because the system evolution takes place in two different coordinate systems defined by T_k and T_{k+1}, respectively. This leads the basis vectors for the initial time step and the final time step to be different. These frames are defined by left and right eigenvectors of the Hankel matrix during the identification of observability and controllability matrices. Following the development in [13], let us consider $\tilde{\hat{A}}_k$ as the linear transformation of $\hat{A}_k$:

$$\begin{aligned} \tilde{\hat{A}}_k &= \hat{\boldsymbol{O}}_k{}^{\dagger} \hat{\boldsymbol{O}}_{k+1} \hat{A}_k \\ &= T_k^{-1} \boldsymbol{O}_k{}^{\dagger} \boldsymbol{O}_{k+1} T_{k+1} T_{k+1}^{-1} A_k T_k \\ &= T_k^{-1} \underbrace{\boldsymbol{O}_k{}^{\dagger} \boldsymbol{O}_{k+1} A_k}_{\tilde{A}_k} T_k \\ &= T_k^{-1} \tilde{A}_k T_k, \end{aligned} \tag{21}$$

where $\boldsymbol{O}_k$ is the observability matrix at time k and by virtue of (21), $\tilde{\hat{A}}_k$ and $\tilde{A}_k$ are now similar matrices, i.e, they share the same eigenvalues. One can utilize the observability matrices corresponding to two different identified realizations of the system matrices to define a common frame to predict system response. In this new frame, the two different realizations are also guaranteed to be similar.

C. Choice of Lifting Functions

One of the core challenge of the Koopman operator theoretic approach is to specify (directly or indirectly through decompositions) the Hilbert space of measurement functions of the state of the system [5]. Indeed, there is no doubt that the ability of an analyst to apprehend the behavior of a dynamical system strongly depends upon the mathematical representation of the physical system, due to the fact that nonlinearity is not an inherent attribute of a physical system, but rather dependent upon the mathematical description of the system's behavior [25]. In other words, identifying the correct measurement functions may not be simple and may become arbitrarily complex once iterated through the dynamics. Oftentimes, a more automatic way of building the Koopman operator is considered by selecting an orthogonal basis. For example, when a polynomial basis is chosen to represent the Koopman operator, then it is closely related to Carleman linearization [26], [27], which has been used extensively in nonlinear system analysis [28]–[31]. An interesting point worth noting is that the first n rows of the Koopman operator corresponds to coefficients of higher-order state-transition matrices. To get an insight into this, let us consider the Taylor series expansion of the flow of nonlinear dynamical system of (22) for zero input, i.e.,

$$\delta x_{a_{k+1}} = \frac{\partial x_{a_{k+1}}}{\partial x_{b_k}} \delta x_{b_k} + \frac{\partial^2 x_{a_{k+1}}}{\partial x_{b_k} \partial x_{c_k}} \delta x_{b_k} \delta x_{c_k} + \ldots, \tag{22}$$

$$a, b, c = 1, 2, \ldots, n \tag{23}$$

Hence, one can directly identify coefficients of aforementioned Taylor series expansion (also known as higher order state transition matrices) when one uses polynomial basis for the lifting process. In general, the upper-left $n \times n$ block matrix will corresponds to the first-order state transition matrix and coefficients in other columns will simply be the exact same coefficients of higher-order state transition tensors, but rearranged:

$$A_k = \begin{bmatrix} \Phi_k^1 & A_k^{(1)} \\ A_k^{(2)} & A_k^{(3)} \end{bmatrix}. \tag{24}$$

$A_k^{(1)}$ will contain coefficients of higher-order tensors while $A_k^{(2)}$ and $A_k^{(3)}$ are sensitivities associated with higher-order polynomial measurements.

Hence, the use of polynomial basis functions in the lifting process helps us in identifying higher order terms in Taylor series expansion of nonlinear flow.

IV. Numerical Examples

This section considers two problems to showcase the utility of the time-varying Koopman operator in predicting the response of a nonlinear system. The first example corresponds to a benchmark problem where a finite dimension Koopman operator can be derived while the second example corresponds to nonlinear oscillator where a finite dimension approximation of infinite dimension Koopman operator is discussed. In both the examples, polynomial basis functions are considered for the lifting process and time-invariant

Koopman operator is also identified to showcase the accuracy gained for the same degree of the lifting process.

A. Single fixed-point Problem

Let us consider a time-varying version of a single fixed-point nonlinear dynamical system introduced in Ref. [6]:

$$\dot{x} = \mu(t)x,\ \dot{y} = \lambda(t)(y - x^2) \tag{25a}$$
$$\mu(t) = -0.5 + 1.5\sin(4\pi),\ \lambda(t) = -0.2 + 0.5\cos(6\pi t)$$

Notice that one can obtain the following analytical expression for the continuous time Koopman operator by appending the state vector with x^2 as measurements, i.e., $\boldsymbol{\psi} = \{x, y, x^2\}$:

$$\mathcal{K}(t) = \begin{bmatrix} \mu(t) & 0 & 0 \\ 0 & \lambda(t) & -\lambda(t) \\ 0 & 0 & 2\mu(t) \end{bmatrix}. \tag{26}$$

Furthermore, one can write:

$$\boldsymbol{z}_{k+1} = A_k \boldsymbol{z}_k,\ \boldsymbol{z}_k = \begin{bmatrix} x_k, & y_k, & x_k^2 \end{bmatrix}^\intercal, \tag{27}$$

For identification purposes, the measurement data is recorded at a frequency of 10 Hz for a time interval of 0-10 seconds. Two random initial conditions drawn from zero mean normal distribution of standard deviation $\sigma_0 = 0.1 I_2$ are used to simulate the response of the nonlinear system for identification purposes. Figure 2 shows the phase plot displaying the true and identified trajectories (from random initial condition with same distribution) while Table I presents the overall root mean squared error (RMSE) for the two predicted trajectories. As expected, the prediction error corresponding to conventional time invariant Koopman operator is several order of magnitude worse than the prediction error for the time varying Koopman operator.

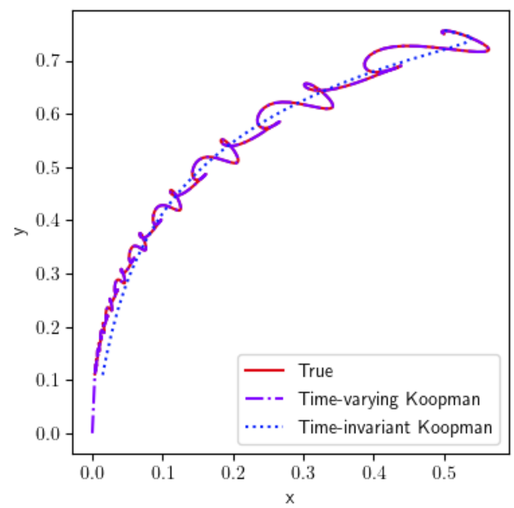

Fig. 2: Phase plot for the single fixed-point problem

TABLE I: RMSE for the trajectories predicted by the two operators

TI Koopman	TV Koopman
$1.2 \cdot 10^{-10}$	$3.5 \cdot 10^{-2}$

Table II presents the RMS error associated with the time evolution of the three discrete-time eigenvalues of true as well as identified A_k matrices after applying for appropriate coordinate transformations. Since eigenvalues match with a very good accuracy, the identified model is not only able to reproduce I/O data of the time-varying original dynamical system but was also able to capture dynamical features embedded in the time-varying state transition matrix.

TABLE II: RMSE for the time-evolution of the TI and TV Koopman Operators eigenvalues

Eigenvalue nº	TI Koopman	TV Koopman
Eigenvalue 1	$1.4 \cdot 10^{-2}$	$5.8 \cdot 10^{-10}$
Eigenvalue 2	$1.5 \cdot 10^{-2}$	$5.4 \cdot 10^{-10}$
Eigenvalue 3	$1.5 \cdot 10^{-2}$	$1.1 \cdot 10^{-9}$

The next example will present a more challenging situation where there is no easy closure for the Koopman operator, hence testing the capabilities of TVERA to provide a finite approximation.

B. Duffing Oscillator with time-varying parameters

The second example corresponds to the nonlinear oscillator known as the Duffing oscillator governed by following equations with time-varying coefficients

$$\dot{x} = y,\ \dot{y} = -\delta(t)y - \alpha(t)x - \beta(t)x^3 + u(t) \tag{28a}$$
$$\delta(t) = 0.2 + 0.1\sin(4\pi t) \tag{28b}$$
$$\alpha(t) = 1 + 0.1\sin(6\pi t + \pi/2) \tag{28c}$$
$$\beta(t) = -1 + 0.1\sin(8\pi t + \pi) \tag{28d}$$

Unlike the previous example, the Koopman operator augmented with polynomial measurements in x and y is of infinite dimension. Three different cases are considered corresponding to different order lifting functions to approximate the true infinite order Koopman operator with finite dimension time invariant as well as time-varying Koopman operators.

1) Case 1: Linear basis functions in x and y.
2) Case 2: Basis function up to degree 2 in x and y.
3) Case 3: Basis functions up to degree 3 in x and y.

The measurement data is recorded at a frequency of 10 Hz for 20 seconds for the simulation purposes. For this representative case, it is desired to identify the time-varying linear departure dynamics from the nominal trajectory. A nominal trajectory with initial condition $\boldsymbol{x}_0 = \begin{bmatrix} 0.1 & -0.2 \end{bmatrix}^\intercal$ and zero input is considered. A true trajectory is simulated by random sampling of initial deviation from a zero mean Gaussian distribution with standard deviation of 0.05 with zero input for identification purposes.

Table III shows the RMS error averaged over 10 random runs for all the three test cases. As expected, the prediction errors corresponding to time-varying Koopman operator are 3-4 orders of magnitude better than prediction errors corresponding to time-invariant Koopman operator. This numerical simulation also confirms that the accuracy of the time-invariant as well as time-varying Koopman operators improves with the increase in lifted degree. Furthermore, the time-varying Koopman operator provides at least two order of magnitude better prediction accuracy than the prediction errors corresponding to conventional time-invariant Koopman operator for all the three test cases. While the accuracy of the time-invariant Koopman operator for lifted degree 3, i.e., test case 3 is comparable to actual linearization of the nonlinear flow, the prediction accuracy corresponding to time-varying Koopman operator is five orders of magnitude better than its time-invariant counterpart for lifted degree 3. These results clearly demonstrates the effectiveness of the time-varying Koopman operator as compared to conventional time-invariant Koopman operator.

TABLE III: RMSE for the Duffing departure trajectories

Lifted space	**TI Koopman**	**TV Koopman**
Case 1	$7.5 \cdot 10^{-3}$	$8.8 \cdot 10^{-6}$
Case 2	$6.9 \cdot 10^{-3}$	$2.4 \cdot 10^{-6}$
Case 3	$8.0 \cdot 10^{-4}$	$9.1 \cdot 10^{-8}$
Actual Linearization		$7.1 \cdot 10^{-4}$

V. CONCLUSION

The concept for time-varying Koopman operator to predict the flow of nonlinear dynamical systems is introduced. The Koopman operator provides an infinite dimension coordinate system in which the flow of the dynamical system is linear. An extension of time invariant subspace methods known as time-varying eigensystem realization algorithm (TVERA) is used to find a finite dimensional approximation of the time-varying Koopman operator. A key advantage of the TVERA formulation is that it also provides a coordinate transformation for linear time-varying systems in which different realizations of the system are similar. Two benchmark problems showcase the utility of the developed approach and its relative merits with regard to conventional time-invariant Koopman operator theory.

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