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SEMI-ANALYTICAL RAPID ORBIT DETERMINATION APPROACH FOR PERTURBED TWO BODY PROBLEM

Erin Cope[*], Roshan Eapen[†], Puneet Singla[‡]

Ensuring flight safety by rapidly determining orbits of observed objects is of paramount importance to maintain the economic value of space assets. While optical sensors routinely provide angles-only data, traditional algorithms for orbit determination depend on methods that approximate the nonlinear dynamics of resident space objects. The goal of this paper is to develop tools to incorporate information from the nonlinear dynamics while keeping the implementation of these orbit determination methods computationally tractable. Specifically, this paper proposes to combine advancements in semi-analytic satellite theory with statistical methods to accurately compute state transition matrices in a Jacobian free manner.

INTRODUCTION

With the increase in number of Resident Space Objects (RSO), the number of close approaches and possibility of conjunction is increasing in the regions extending from LEO to GEO.[1] Determining the orbit of satellites using current techniques such as an initial orbit determination, batch least-squares, Kalman filtering, etc. presents a number of unique challenges with respect to acceleration of the algorithm.[2,3] An important aspect in the orbit determination approach that has to be taken into account is that: *"... not just any prediction model will suffice... The NORAD element sets must be used with one of the models described in this report in order to retain maximum prediction accuracy*.[3] This implies that the orbit determination strategy must be able to use NORAD element sets, and must be able to update the NORAD element sets in the space catalog.

The traditional approaches to orbit determination approximate the complex dynamics and highly non-linear nature of RSO dynamics. Additionally, dynamical models, and their analytical expressions (such as the Jacobian, the state transition matrix, etc.) are difficult to update or append when additional perturbations are included in an existing model. This means that they are seldom used in conjunction with new or different perturbation models without significant changes to the existing model itself. Under the hyperspace challenge, a topic area of interest is the rapid trajectory tracking of space objects.[4] Under its purview,

[*]Graduate Student, Department of Aerospace Engineering, The Pennsylvania State University, University Park, PA, 16802

[†]Assistant Professor, Department of Aerospace Engineering, The Pennsylvania State University, University Park, PA, 16802

[‡]Professor, AIAA Associate Fellow, AAS Fellow, Department of Aerospace Engineering, The Pennsylvania State University, University Park, PA, 16802

statistical algorithms are sought which produce estimates of satellite states ***without*** approximating the nonlinear dynamics and maintaining computational tractability.

One hurdle to this approach is that the measurements are often the position vector of the Resident Space Object (RSO), or angles and angle-rates from an observation site. The transformation map from NORAD element sets (often expressed in Brouwer's mean elements) to the measurement space is a highly nonlinear one. Add to this further obstacles from optical sensors such as the poor observability of the radial component of the RSO population. Additionally, RSOs in the LEO regions, as well as GEO regions are often maneuvering to perform station-keeping, or to avoid collisions. As the number of RSO continue to increase the demand for data also increases while there continues to be time constraints on observation sites. This increase in demand and reduced observations make the traditional orbit determination methods less accurate, and the rapidity of the orbit determination methods more important.

This paper aims to develop tools to incorporate stastistical information from position measurements to inform the orbit determination process. In fact, one can envision the orbit determination process as a multi-dimensional Newton-Raphson root solving method of $y = f(x)$ with a least-squares statistical treatment of the "known data" (y).[3] To this effect, an orbit determination methodology is developed that uses stochastic linearization to efficiently circumvent the computation of the state transition matrix and directly obtain the transformation between the measurement space and the NORAD element set. Note that is is inherently a highly nonlinear problem which requires the transformation of Brouwer's mean elements, to osculating elements, and then to measurement space variables. One advantage of this method is that it would also enable the rapid update of the TLE data for the tracked RSO. Using statistical methods in conjunction with analytical orbit theories renders the linearization fairly accurate over a larger domain.

The remainder of the paper is organized as follows. In the following section, the formulation of the problem is discussed including the orbit determination process and where this work's improvements to that process will be made, namely in the use of statistical linearization to compute a direct mapping from the measurement space to the state space. Next, the solution methodology is covered. This methodology includes the Simplified General Perturbation (SGP4) model which is used to analytically propagate the state, statistical linearization to compute first order linear mappings from the measurement to the state space, the use of the conjugate unscented transformation as a non-product quadrature scheme, and the regression method of non-linear least squares used for the initial orbit determination. The next two section shows the results of our work and discussions on the methodology. This includes a Monte-Carlo simulation for the CUT method and the convergence results from the NLS method that utilizes the direct mapping created. This is followed by the summary of the work completed in this paper.

PROBLEM FORMULATION

This paper presents a novel way to integrate statistical analysis and deterministic tools to efficiently implement an orbit determination method is a fast and computationally tractable

manner. Among the different orbit determination methods available, this paper utilizes a non-linear least squares routine to perform orbit fitting. The least-squares methods provide attractive properties such as simplicity, stability, and speed. The dynamics model is a big part of the orbit determination pipeline. In this work, the Simplifies General Perturbations SGP4 suite is used to propagate the states. The primary reason for using SGP4 is because it is an analytic propagator compatible with the NORAD element sets.

The dynamics of the orbital motion will be assumed to be deterministic, and without noise. Deterministic dynamics allows for the expression of the State Transition Matrix (STM) to be obtained as $\delta\mathbf{x}(t) = \Phi(t, t_0)\delta\mathbf{x}_0$ to be valid, where $\delta\mathbf{x}(t)$ is some small variation to the state. The STM can be used to describe how the perturbed states evolve over time along a state trajectory, and can also be used to measure sensitivities of current states with respect to initial states. For certain variations of the perturbed two body problem there is an analytical solution to the STM, however those solutions do not consider drag, which we will consider. More discussion on the specific perturbations considered are discussed in Table 1.In this work, we will utilize the semi-analytical orbit propagation theory in conjunction with statistical methods to accurately compute the STM in a Jacobian free manner. The standard outline of the orbit determination routine is provided in the Figure 1.

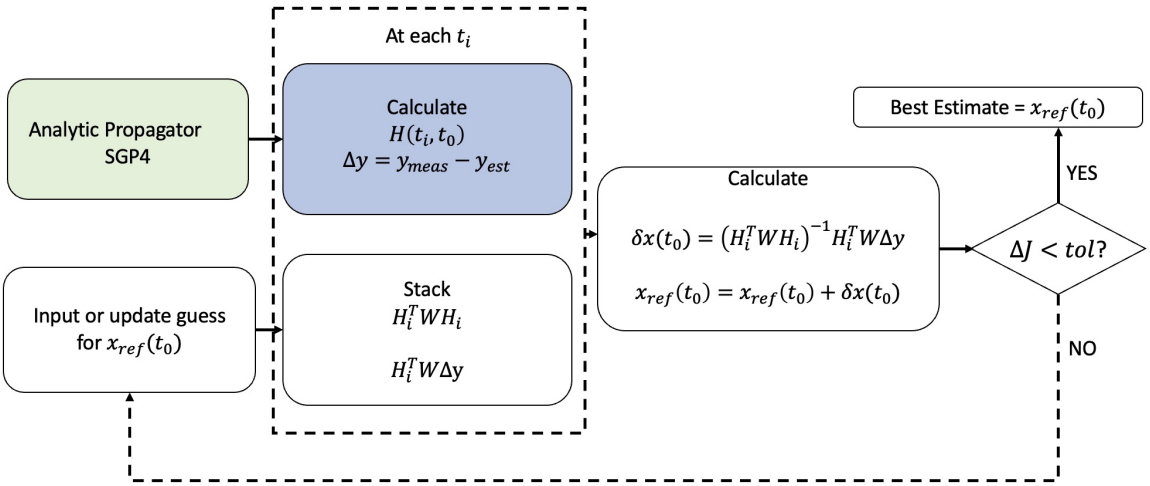


Figure 1: Iterative Nonlinear Least-Squares for Orbit Determination with this Work's Updates in Highlighted Boxes

This is an iterative procedure, with the initial guess provided through an Initial Orbit Determination method, or through a previously made estimate such as data from a two line element (TLE) set. This paper proposes two areas in the orbit determination pipeline to enable its rapid implementation:

1. The dynamics propagation (shown in the green box in Fig. 1) is done using an analytic orbit propagator, SGP4.[5,6] The SGP4 routine can analytically propagate an

initial set of mean elements to the state vector $\mathbf{x}(t) = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$ at a future epoch.

2. The blue box to obtain the mapping, H, between the state space and the measurement space using statistical linearization. In contrast to conventional algorithms based upon linear theory, statistical linearization usually provides good approximation over a region of interest even in the presence of strong nonlinearities.

The primary contribution of this paper will be the accelerations in the least squares routine primarily due to the modifications in the two shaded blocks shown in Fig. 1. The analytic nature of our propagation accompanied by the stochastic linearization technique removes numerical dependencies, which can result in decreased computational time. Furthermore, the STM computed through stochastic linearization will be valid over a larger domain than a numerically propagated STM, thereby not requiring a recomputation (and re-propagation) at every iteration of the batch least squares pipeline.

The methodology presented in this paper contrasts that done by the SGP4 orbit determination pipeline where the STM is obtained using finite (or central) differences, or analytical partials.[3] Both approaches used therein suffer from a smaller domain of validity and/or computational tractability. Elegant operational systems generally employ the numerical integration approach, however, with a simple analytical propagation theory (SGP4) it may make sense to simply utilize it for statistical computations. The computational advantage from using analytic partials is often negated by simply allowing a few more routine calls of SGP4.[3] In addition, the partial derivatives for each force model are not needed as they are all included in the function call of SGP4. The implementation of a streamlined statistical linearization method as well as other mathematical approaches taken are discussed in the following section.

THE RAPID STATISTICAL ORBIT DETERMINATION PIPELINE

The primary contribution of this paper is the implementation of a statistical linearization scheme to obtain a direct transformation from the measurement space to the space of NORAD element sets. The method of statistical linearization has proved to be one of the most useful approximate technique in various applications.[7] However, one usually needs to rely on sampling based methods such as Monte Carlo methods to compute the statistical linearization approximation. This makes the method computationally intractable as the system dimension is increased. To alleviate this problem, recently developed Conjugate Unscented Transform (CUT) method is used to perform an optimum sampling. This optimal sampling provides the STM in a semi-analytic manner, leveraging the simplicity of polynomials and their derivatives. We also apply the method of non-linear least squares to find a solution to our overdetermined system, and preform orbit determination. The following sections will describe the key elements of the rapid OD pipeline developed. In the following section the propagation method SGP4 will be discussed. It was chosen as the analytical propagation method because it allows the state to be propagated into the future

without the need for numerical integration and utilizes Brouwer's mean elements which can be conveniently found in a satellite's Two Line Element (TLE).

Simplified General Perturbation (SGP4) Model

Simplified General Perturbation 4 (SGP4) Model is an analytic propagation method based off of Brouwer's Theory, which uses Brouwer's mean orbital elements. These mean elements are the state elements used in this paper. These are found by averaging the osculating orbital elements first over mean anomaly, the fast angle, then over the argument of perigee, the slow angle.[8] Since SGP4 deals with mean elements, it is simple to pull the necessary orbit information from a TLE and put it in a form for SGP4 to use. However, while SGP4 is based on Brouwer's Theory, the mean motion listed in the TLE is based on Kozai's definition.[3,9] Table 1 below shows the perturbations considered by SGP4 based on orbit characteristics.[10]

Table 1: Perturbations Considered by SGP4 Based on Orbit Parameters

Orbit Period	Perturbations considered
$\mathcal{P} < 225$	J_2, J_2^2, J_3, J_4, drag
$\mathcal{P} \geq 225$	J_2, J_2^2, J_3, J_4, drag, lunisolar
$680 < \mathcal{P} < 760$ and $e > 0.5$	J_2, J_2^2, $J_{2,2}$, J_3, $J_{3,2}$, J_4, $J_{4,4}$, $J_{5,2}$, $J_{5,4}$, drag, lunisolar
$1200 < \mathcal{P} < 1800$	J_2, J_2^2, $J_{2,2}$, $J_{3,1}$, $J_{3,2}$, J_4, drag, lunisolar

Traditionally, in order to relate Brouwer's mean elements to measurement data, often range, azimuth, and elevation observations, the elements would need to first be transformed from mean to osculating, then to position and velocity, and finally to the measurement state, as illustrated in Fig. 2.

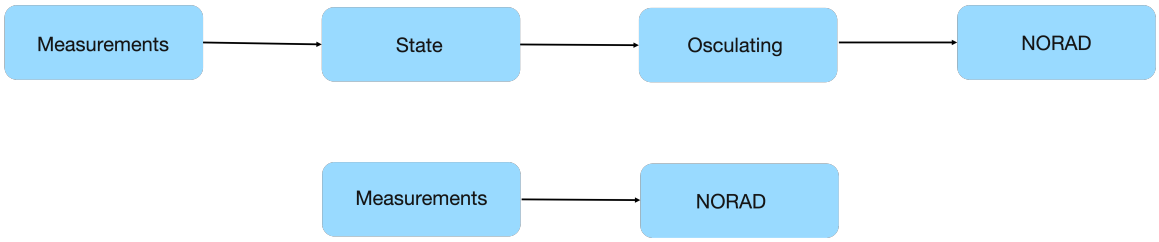


Figure 2: Typical Element Conversion from Measurements to Catalog Update

Since the mapping matrix (H) is created by approximating the measurement model using basis functions, this approach allows us to directly relate Brouwer's mean elements to measurement data.

Statistical Linearization method to Compute First Order STMs

Let us consider the Taylor series expansion of the departure motion from the nominal trajectory:

$$\delta\mathbf{x}(t) \approx \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_n=0}^{\infty} \frac{\delta x_{0_1}^{N_1}\delta x_{0_2}^{N_2}\cdots\delta x_{0_n}^{N_n}}{N_1!N_2!\cdots N_n!}\frac{\partial^{N_1+N_2+\cdots+N_n}}{\partial x_{0_1}^{N_1}\partial x_{0_2}^{N_2}\cdots\partial x_{0_n}^{N_n}}\psi(t,\mathbf{x}_0), \qquad (1)$$
$$N_1 = N_2 = \cdots N_n \neq 0$$

In other words, one can expand $\delta\mathbf{x}(t)$ in terms of polynomial basis functions:

$$\delta\mathbf{x}(t) \approx \sum_{i=1}^{m} \boldsymbol{c}_i(t)p_i(\delta\mathbf{x}_0) = \mathbf{c}(t)\boldsymbol{p}(\delta\mathbf{x}_0) \qquad (2)$$

where $p_i(\delta\mathbf{x}_0)$ is the polynomial function of total degree i, $\mathbf{c}(t)$ is a matrix and $\boldsymbol{p}(\delta\mathbf{x}_0)$ is a vector. Note that the coefficients of the linear terms corresponds to the conventional state transition matrix, which is valid only in the neighborhood of the nominal trajectory denoted by $\mathbf{x}(t)$. Higher order terms can be considered as higher order state transition matrices.[11–13] If initial condition, $\mathbf{x}_0$ is a random variable with prescribed density function $\rho(\mathbf{x}_0)$, then it would make sense to compute the first order and higher order state transition matrix valid over the domain of initial condition uncertainty. In this respect, one can pose the following problem (also known as statistical linearization) to compute state transition matrix equivalent coefficients, $c_i(t)$:

$$\min_{c_i(t)} J = \frac{1}{2}\int \left(\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)\right)^T \left(\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)\right)\rho(\mathbf{x}_0)d\delta\mathbf{x}_0 \qquad (3)$$

$$= \frac{1}{2}\langle(\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)), (\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0))\rangle \qquad (4)$$

where the angle brackets are generalization of inner products. Taking the first derivative of J with respect to coefficient vector $\mathbf{c}$ leads to the following normal equations to solve for the coefficients, $\mathbf{c}$:

$$\sum_{j=1}^{m}\langle p_i(\delta\mathbf{x}_0), p_j(\delta\mathbf{x}_0)\rangle c_j = \langle\delta\mathbf{x}(t), p_i(\delta\mathbf{x}_0)\rangle, \quad i = 1, 2, \cdots, m \qquad (5)$$

This can be written in a compact form as:

$$\mathbf{M}(t)\mathbf{c}(t) = \mathbf{b}(t), \;\; M_{ij}(t) = \langle p_i(\delta\mathbf{x}_0), p_j(\delta\mathbf{x}_0)\rangle, \; b_i(t) = \langle\delta\mathbf{x}(t), p_i(\delta\mathbf{x}_0)\rangle \qquad (6)$$

To this end, $\delta\mathbf{x}_0$ is assumed to be a function of a standardized random vector, $\boldsymbol{\xi}$ defined by a standardized density function, $\rho(\boldsymbol{\xi})$. If $\delta\mathbf{x}_0$ is assumed to be Gaussian random vector with

a prescribed mean vector, $\boldsymbol{\mu}$ and a covariance matrix, $\boldsymbol{\Sigma}$, then $\boldsymbol{\xi}$ can be a vector of Gaussian random variables with zero mean and identity covariance. Hence, $\delta\mathbf{x}_0$ can be written as:

$$\delta\mathbf{x}_0 = \mathbf{a}_0 + \mathbf{a}_1\boldsymbol{\xi} \tag{7}$$

where, $\mathbf{a}_0 = \boldsymbol{\mu}$ and $\mathbf{a}_1 = \sqrt{\boldsymbol{\Sigma}}$. Hence, basis function, p_i can be defined as a function of $\boldsymbol{\xi}$ rather than $\delta\mathbf{x}_0$. Furthermore, if one chooses polynomial basis functions to be orthogonal polynomials associated with the assumed probability distribution for the input variables, $\boldsymbol{\xi}$, then $\mathbf{M}$ will be a diagonal matrix and Eq. (6) can be re-written as:

$$\mathbf{M}(t)\mathbf{c}(t) = \mathbf{b}(t), \;\; M_{ii}(t) = \langle p_i(\boldsymbol{\xi}), p_i(\boldsymbol{\xi})\rangle, \; b_i(t) = \langle \delta\mathbf{x}(t,\boldsymbol{\xi}), p_i(\boldsymbol{\xi})\rangle \tag{8}$$

These orthogonal polynomials can be computed through the application of the Gram-Schmidt orthogonalization process. The major challenge lies in computing the multi-dimensional expectation integrals, which appear in the expression for $b_i(t)$. Generally, these integrals are evaluated numerically, i.e.,

$$b_i(t) = \langle \delta\mathbf{x}(t), p_i(\boldsymbol{\xi})\rangle = \sum_{i=1}^{N} w_i p_i(\boldsymbol{\xi}_i)\delta\mathbf{x}(t,\boldsymbol{\xi}_i) \tag{9}$$

The mapping matrix (H) can now be written as:

$$H = \frac{\delta\mathbf{y}}{\delta\boldsymbol{\xi}}\frac{\delta\boldsymbol{\xi}}{\delta\mathbf{x_0}} = \sum c_i^k \frac{\delta\phi_i}{\delta\boldsymbol{\xi}}\frac{\delta\boldsymbol{\xi}}{\delta\mathbf{x_0}} \tag{10}$$

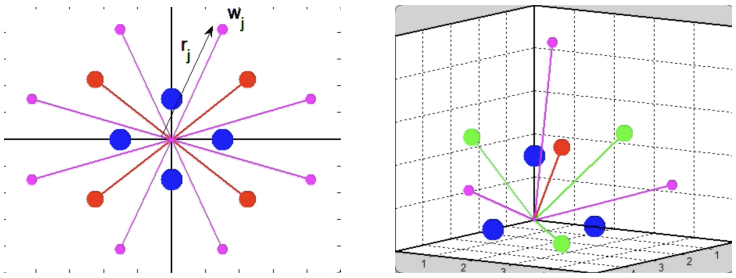
Recall from Eq.(15) that $\boldsymbol{\xi}$ is a scaled variable where $\boldsymbol{\xi} \in [-1, 1]$ the derivative with respect to the initial state variable comes from the scaling factor $\mathbf{a}_1$.

While statistical linearization is a powerful tool, it is not often used in the context of STM approximation because it generally requires a large number of points to solve the expectation integrals, which no longer makes it a rapid technique. However, Conjugate Unscented Transform (CUT) is a quadrature method that will allow the evaluation of these integrals in far fewer points. This technique is discussed in the following section.

Conjugate Unscented Transformation

In this work, we will utilize non-product quadrature scheme known as the Conjugate Unscented Transform (CUT)[14] method in conjunction with the concept of statistical linearization to compute STM in a derivative free and a computationally attractive manner. The computed mapping is valid over the desired domain represented by a probability density function rather than valid along a single trajectory of a dynamical system. The CUT method provides minimal non-product quadrature rules to compute the multi-dimension expectation integrals involving Gaussian and uniform density functions. Rather than using tensor products as in the Gaussian quadrature methods,[15] the CUT approach judiciously selects special structures of symmetric points. These new sets of points are guaranteed to exactly evaluate expectation integrals involving polynomial functions with significantly

fewer points. The CUT approach has been used for many filtering and control applications[16–19] and in particular it was utilized for accurate conjunction analysis between two space objects when the PDF for orbital state vectors are significantly non-Gaussian in nature.[19] While MC methods generally suffer from slow convergence rates, the importance sampling strategies to alleviate this problem (e.g., Markov Chain MC) cannot be parallelized effectively. An alternative to the random sampling is the quadrature scheme, such as the popular Gaussian Quadrature, which involves deterministic points carefully chosen to reproduce exactly the integrals for polynomials, i.e., moments of the density function. The Gaussian quadrature schemes exactly reproduce the integral of a polynomial of degree $2M-1$ with M^m points in a m-dimension space. Even for a moderate-dimension system involving, say, 6 random variables, the number of points required to evaluate the expectation integral with only 5 points along each direction is $5^6 = 15,625$. The sparse grid quadratures, and in particular Smolyak quadrature, take the sparse product of one dimensional quadrature rules and thus have fewer points than the equivalent Gaussian quadrature rules, but at the cost of introducing negative weights.[20] Fortunately, the Gaussian quadrature rule is not minimal for $m \geq 2$, and there exists quadrature rules requiring fewer points in high dimensions.[15] For example, the Unscented Transformation (UT)[21] is exact to degree 2 but with linear growth of points with dimension. However, the UT cannot be used to reproduce higher order moments.



(a) CUT Points in 2D (b) CUT Points in 3D

Figure 3: A Schematic of CUT Axes

Previously, a non-product quadrature rule known as the Conjugate Unscented Transformation (CUT)[16–19] has been developed. The CUT approach can be considered an extension of the conventional UT method that satisfies additional higher order moment constraints. Rather than using tensor products as in Gauss quadrature, the CUT approach judiciously selects special structures to extract symmetric quadrature points constrained to lie on specially defined axes as shown in Figure 8. For each cubature point, two unknown variables, a weight w_i and a scaling parameter r_i are assigned. The moment constraints equations for the desired order are derived in terms of unknown variables r_i and w_i. Because of the symmetries of cubature points, the odd-order moment constraints equations are automatically satisfied, so the w_i and r_i are found by solving just the even order equations. The order of these moment constraint equations dictates the set of cubature points. These new sets of so-called sigma points are guaranteed to exactly evaluate expectation integrals involving

polynomial functions with significantly fewer points. Figure 4 shows a comparison of the number of points required for CUT and Gauss-Legendre quadratures for similar accuracy, clearly illustrating the reduced growth exhibited by the CUT method. More details about the CUT methodology and its comparison with conventional quadrature rules can be found in Ref.[16–19,22] With the application of the CUT approach, we can generate higher order statistical equivalent transition matrices in a computationally efficient manner.

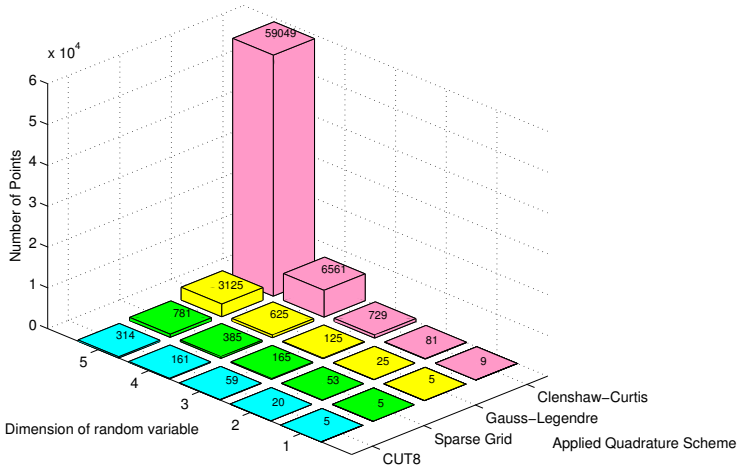


Figure 4: Comparison of Points - 9^{th} Order Accuracy.

The next section will discuss the regression method of nonlinear least squares and its application to the orbit determination process.

Nonlinear Least Squares

Nonlinear Least Squares (NLS) is an regression approach that finds a solution to an overdetemined system by minimizing the square of the residuals. This cost function is shown below where $\hat{y}$ are the measurement values and y are the estimated values.

$$J = (\hat{\mathbf{y}} - \mathbf{y})^T(\hat{\mathbf{y}} - \mathbf{y}) = \Delta\mathbf{y}^T\Delta\mathbf{y} \tag{11}$$

Starting with an initial guess, the estimated state is propagated forward in time to obtain the estimated measurement values, then cost function is calculated. A correction to the state is computed using Eq. (16). This correction is then applied to the state guess and the process is repeated until the change in the cost function between iterations falls below a predetermined tolerance value, where it is considered to have converged. However, convergence is not guaranteed through NLS. Many factors affect this including the accuracy of the initial guess as well as the fidelity of the model.

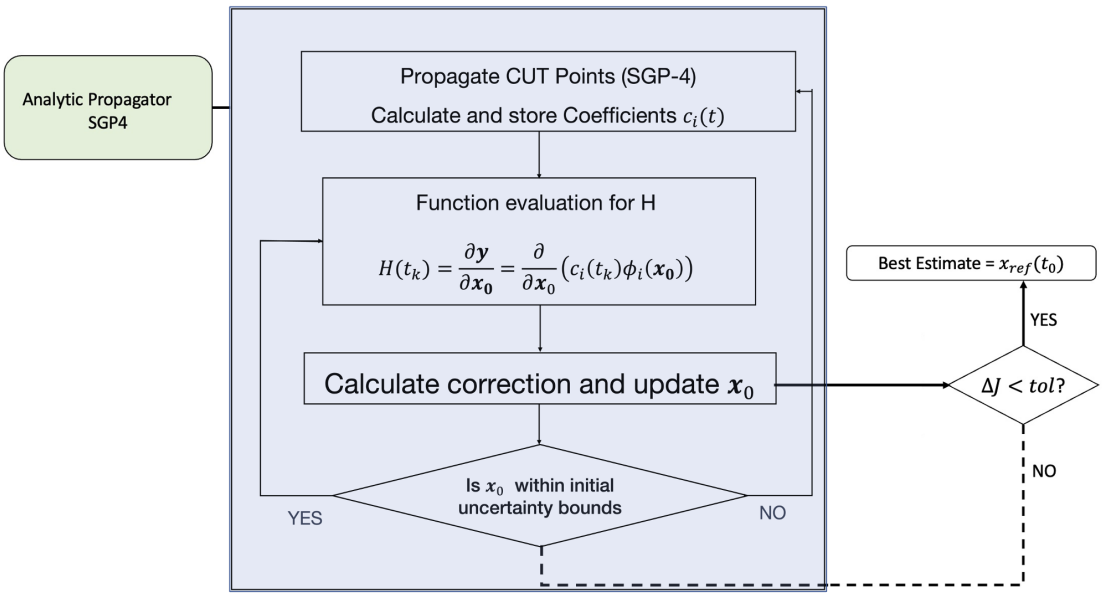


Figure 5: Applied Non-Linear Least Squares Method

All of the mathematical methods discussed in this section were applied to our work, the results of which are discussed in the following section. To analyze the accuracy of the CUT estimation method, a Monte-Carlo simulation was used to analyze error, and the regression model was applied to orbits of varying types, including Low-Earth Orbit (LEO), Medium-Earth Orbit (MEO), Geosynchronous Orbit (GEO), and a Highly Elliptic molniya Orbit (HEO). These four orbits provided the nominal values for the orbit epoch and were used to create the measurement data used for our evaluation.

NUMERICAL RESULTS

The following section will discuss the results of this work, including an evaluation of the accuracy of the CUT method as compared to a Monte-Carlo Simulation, as well as data from a regression analysis.

Model Validation Using Monte-Carlo Simulation

To validate the model, the following satellites were used and their TLE's taken as the nominal orbit for each of the listed orbit types.

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VANGUARD 2          - LEO
GPS BIIF-2          - MEO
AEHF-3 (USA 246)    - GEO
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To test the accuracy of the estimated quadrature model, a Monte-Carlo simulation was used across the bounds of $\boldsymbol{\xi}$. The necessary CUT points were determined by the size of the state dimension and the order of the approximation model. With 6 states and a 4th order polynomial estimation, there are 210 basis function combinations and 301 CUT points. TLE mean elements were used as the nominal value of the state. These CUT points were then scaled up using the a_0 and a_1 so that the CUT points replaced the nominal mean elements. These scaled up CUT points were propagated into the future using SGP4 to create a set of future CUT elements. These future CUT elements were used to find the coefficients for the basis functions. As in Eq. (9), the sum of the products of the weights, basis function, and CUT orbital elements were used to solve for b_i. Then the coefficients were found by solving Eq. (8), where M is the matrix of inner products of the basis functions.

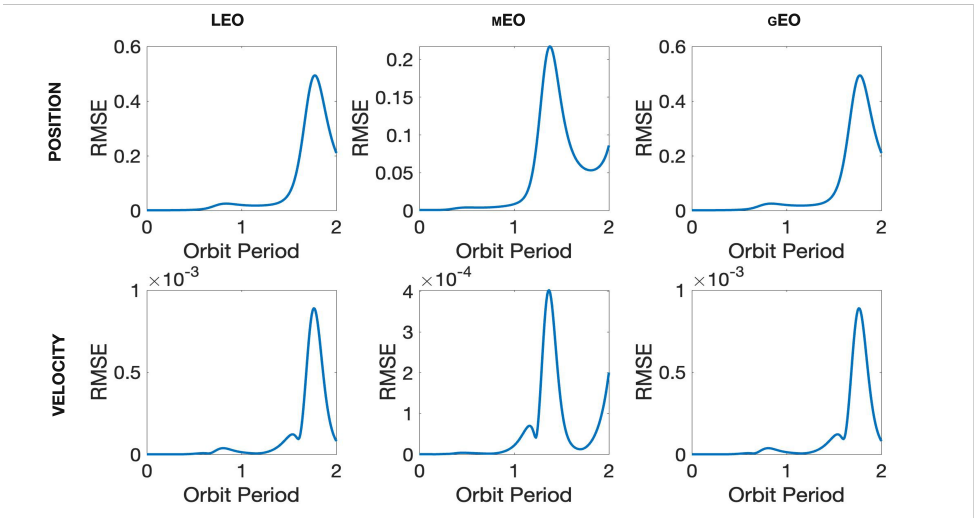


Figure 6: RMSE of Comparison of CUT and Monte Carlo Points Over Two Orbit Periods

Once the coefficients were created for each time step, the MC analysis begins where 1000 random samples on the domain of ξ were created for each element. These MC element sets were propagated into the future at the same time instances as the CUT elements. At each time instance an estimated orbit state was calculated using the basis functions and CUT derived coefficients. Figure 6 and figure 7 show the results of the Root Mean Square Error (RMSE) and Relative Square Error (RSE), respectively, for each LEO, MEO, and GEO orbit considered, over two time periods for each orbit.

The root mean square error in position over two time periods did not exceed 1 km for any orbit, and the error in velocity was of the order of less than a meter per second. These results support the accuracy of the coefficients to estimate the orbit. The relative error also supports the accuracy of the CUT points and coefficients as the error is all of the order 10^{-5}.

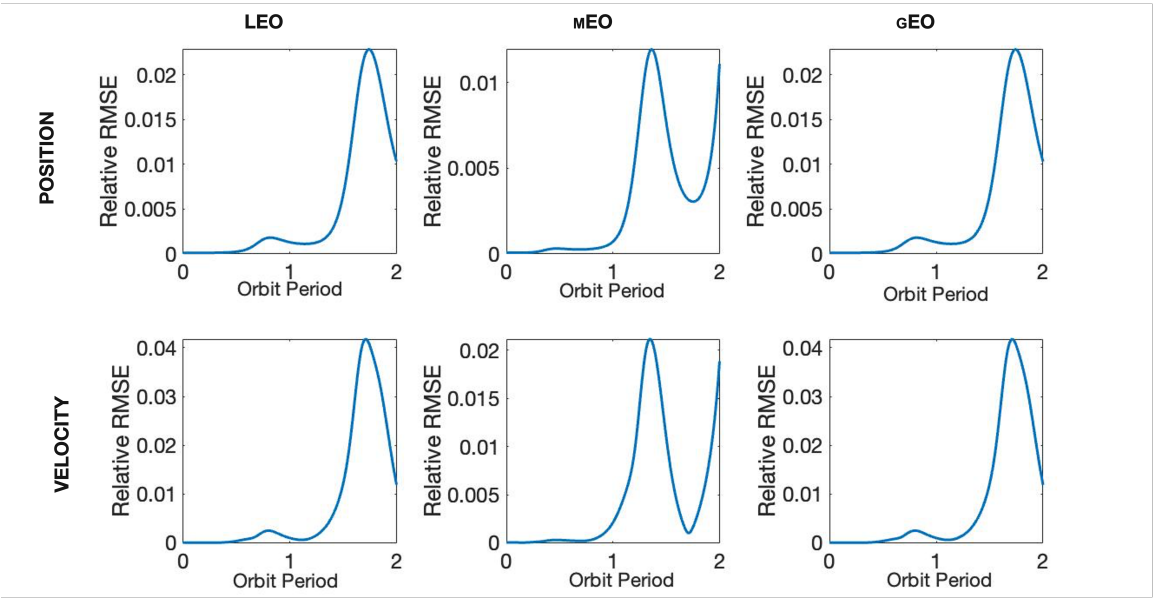
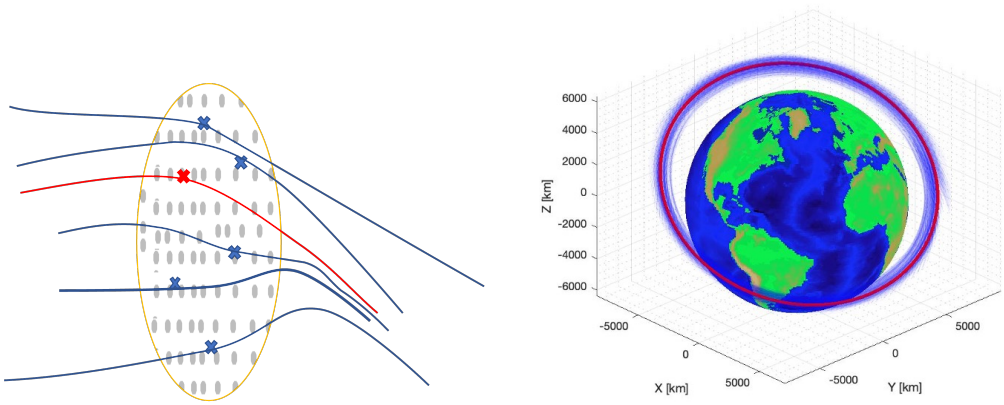


Figure 7: RSE of Comparison of CUT and Monte Carlo Points Over Two Orbit Periods



(a) CUT Domain Schematic (b) CUT Domain for Nominal LEO

Figure 8: CUT Domain Example for Low Earth Orbit

Additionally, the time to solve for all of the coefficients for an entire set of basis functions and CUT points at a given time step is about 0.0124 seconds (run on a macOS Laptop with Apple M1 chip). This allows for coefficients for many time steps to be quickly calculated without adding significant time constraints. Additionally, this code can be parallelized to increase rapidity. After comparing our model to the evaluation with Monte-Carlo points, we applied regression analysis, which is discussed in the next section.

Data from Regression Analysis

The process for computing the coefficients to use in the nonlinear least squares regression is the same for that of the Monte Carlo analysis, however only the coefficients relating to the

position measurements were kept. After the coefficients were created and the measurement data was made by propagating the nominal mean elements with SGP4 to get a position vector and then adding Gaussian noise to the order of 10^{-2}. Once those were completed the least squares process started. For this process the measurements were position in the x, y, and z direction in the ECI frame. The H matrix is created by finding the derivative of the basis functions for a given value of the estimated orbit parameters. Note that these derivatives are with respect to the scaled variable ξ, therefore the estimated parameters used to calculate the derivative values must also be of the scale of ξ.

Table 2: Cost Function Value v. Number of Iterations

Orbit Type	Mean Orbital Elements	a	e	i	Ω	ω	M
LEO	Nominal	8121.58	0.14670	0.57386	1.1297	4.3268	1.6749
	Final Estimation	8121.71	0.14668	0.57386	1.1297	4.3267	1.6750
MEO	Nominal	26560.2	0.01184	0.98824	0.4779	0.9212	5.2926
	Final Estimation	26559.9	0.01184	0.98824	0.4779	0.9204	5.2934
GEO	Nominal	42166.3	0.00507	0.03099	1.2563	3.1461	2.9679
	Final Estimation	42166.9	0.00509	0.03096	1.2558	3.1429	2.9717
HEO	Nominal	26438.8	0.75326	1.09740	0.4500	4.7287	0.2184
	Final Estimation	26438.9	0.75325	1.09740	0.4500	4.7287	0.2185

For each time step the coefficients related to that time step were multiplied with the derivative of the basis functions and then scaled by $\frac{1}{a_1}$ to form the H matrix at the first time step. This process is repeated for each time instance until the entirety of the H matrix is formed. The estimated orbit is also propagated into the future at each time instance and a new matrix of estimated measurements is created. From here the rest of the non-linear least squares process is the same. Next we find $\Delta\mathbf{y}$ (Eq. 11), compute cost (J), and correct the state ($\mathbf{x}_0$) again, if applicable. Figure 9 shows the change in the cost function over each iteration for every orbit type considered.

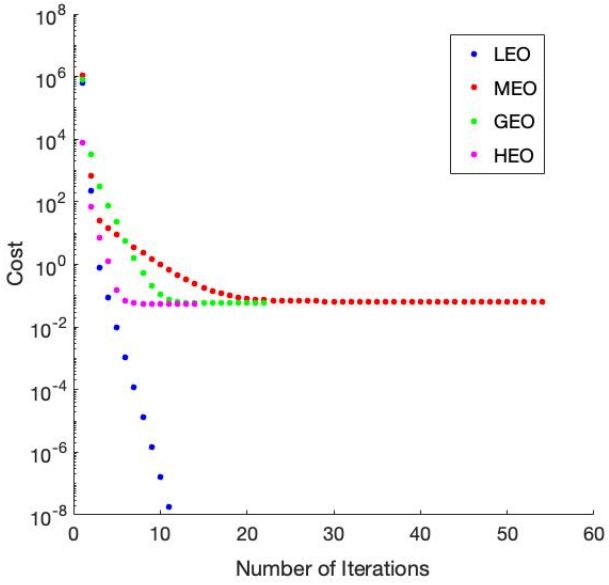


Figure 9: Cost Function Value v. Number of Iterations

DISCUSSION

For a each orbit type whose initial mean orbital elements are shown in table 1, the solver converged every time under the following conditions. Firstly, the initial uncertainty was taken to be $\pm 5\%$. The Measurement data was formed by propagating the nominal mean orbital elements, taking the position values only of those measurements in the ECI frame, and adding noise of the order 10^{-2}. While this was the measurement data for this preliminary work, the measurement type could be changed without change to the problem solution process. The initial guess of the scaled variable ξ, whose true nominal value is a 6 dimensional vector of zeros, was taken to be a random guess of the order of 10^{-2}. Figures 10 through 13 show the overlay of the measurement data in black and the final estimated orbit in red. Figure 10 shows the comparison of the LEO orbits, where the initial uncertainty is $\pm 5\%$, the nominal orbital elements are $\big[8121.58,\ \ 0.14670,\ \ 0.57386,\ \ 1.1297,\ \ 4.3268,\ \ 1.6749\big]$, and which converged in 10 iterations.

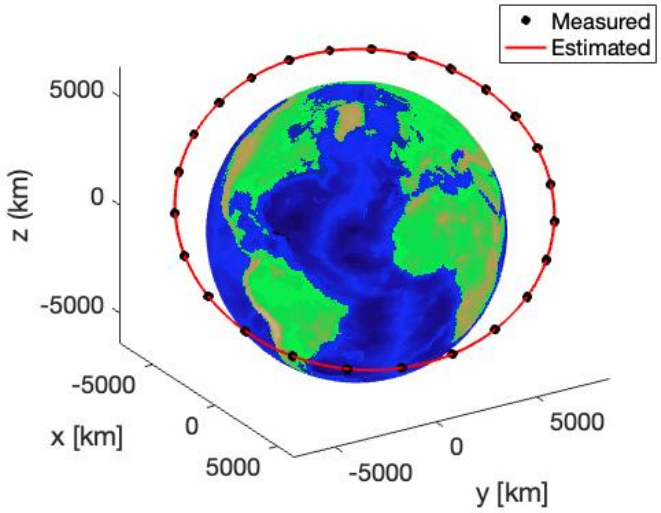


Figure 10: Measured v. Estimated LEO Orbit

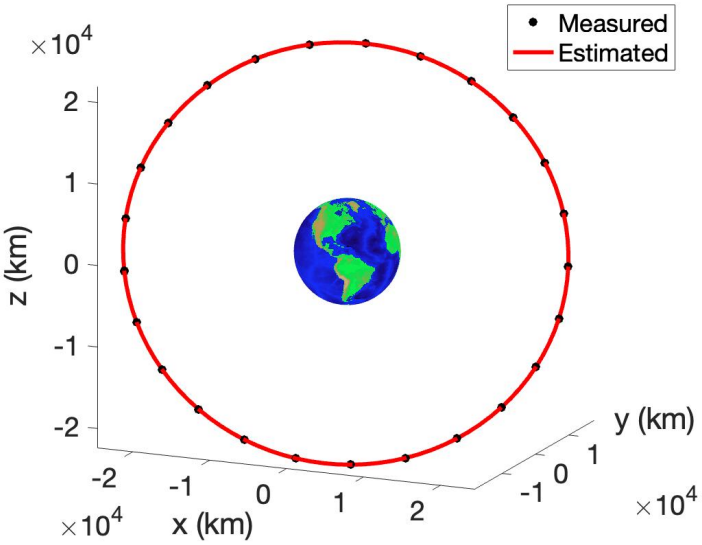


Figure 11: Measured v. Estimated MEO Orbit

Figure 11 shows the comparison of the MEO orbits, where the initial uncertainty is $\pm 5\%$, the nominal orbital elements are $\begin{bmatrix} 26560.2, & 0.01184, & 0.98824, & 0.4779, & 0.9212, & 5.2926 \end{bmatrix}$, and which converged in 10 iterations.

Figure 12 shows the comparison of the GEO orbits, where the initial uncertainty is $\pm 5\%$, the nominal orbital elements are $\begin{bmatrix} 42166.3, & 0.00507, & 0.03099, & 1.2563, & 3.1461, & 2.9679 \end{bmatrix}$, and which converged in 10 iterations.

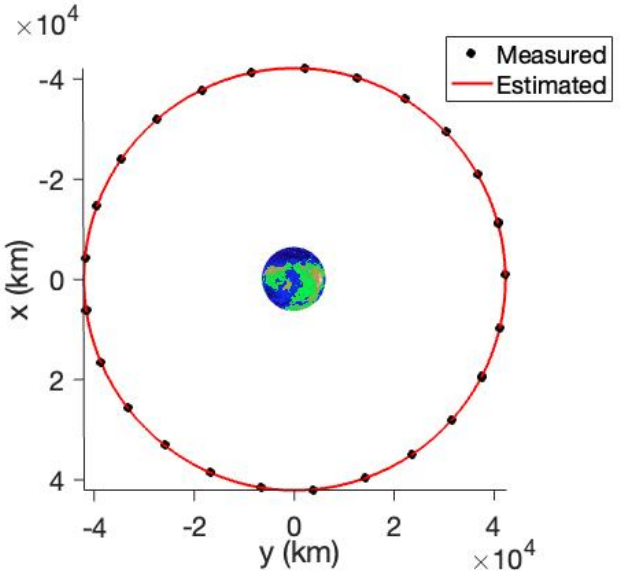


Figure 12: Measured v. Estimated GEO Orbit

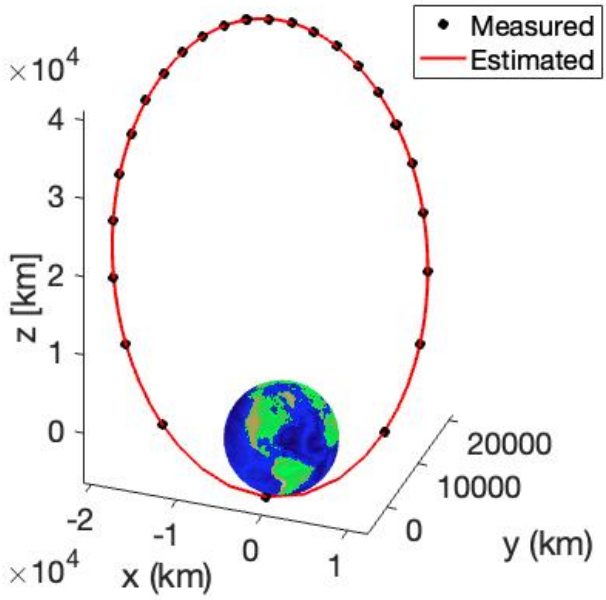


Figure 13: Measured v. Estimated HEO Orbit

Figure 13 shows the comparison of the HEO orbits, where the initial uncertainty is $\pm 5\%$, the nominal orbital elements are $\begin{bmatrix} 26438.8, & 0.75326, & 1.09740, & 0.4500, & 4.7287, & 0.2184 \end{bmatrix}$, and which converged in 10 iterations.

One difficulty with this method of regression is that since the basis are computed using a scaled variable $\boldsymbol{\xi} \in [-1, 1]$, when an initial variable is small the range will be very small as well as the scaling factor $\mathbf{a_1}$. This can cause the scaled variable to quickly go out of bounds,

limiting the ability of the model to correct the guess to the converged value. This is likely to happen for circular or near circular orbits, as even if the eccentricity bounds is scaled to be $\pm 5\%$, the the domain the eccentricity value can fall within during the regression is $\ll 1$. For highly eccentric orbits this is not a problem and the solution will likely converge within just a few iterations, but for geosynchronous and other near circular orbits other steps must be taken to improve the likelihood of convergence. For all of the cases considered in this paper, the CUT coefficients only had to be computed once.

SUMMARY

In this paper we have discussed the need for improved rapidity in orbit determination due to the increasing number of RSO, and the ways in which traditional methods of orbit determination are cumbersome to compute due to the high non-linearity of the dynamics and the models not lending themselves to being easily updated for additional dynamics or different perturbations. Instead of a numerical approach, we took a semi-analytic approach to this problem by applying statistical linearization to create a direct mapping from the measurement space to the state space to circumvent the need for large, difficult to obtain partials within the Jacobian and the need to numerically propagate an STM, allowing for faster computation times and less computational cost. This was done using a non-product quadrature estimation method, CUT. These CUT points were used to accurately and quickly evaluate quadrature estimations of complex, highly nonlinear dynamical systems models, which we applied to our model for orbit determination. The method of propagation was SGP4, an analytic propagator that utilizes Brouwer's mean elements as opposed to osculating orbital elements or other types of measurement data such angles-only or position and velocity. Non-linear least squares was used to determine the estimated value at epoch. This process was able to accurately estimate the highly non-linear dynamics and successfully create a mapping from the measurement space to the state space.

To perform the quadrature estimation for a 6 dimensional state vector at a 4^{th} order approximation, 301 CUT points were used over 210 basis functions. These CUT points were then scaled from the bounds of $\boldsymbol{\xi} \in [-1, 1]$ to a within a 10% uncertainty range around the nominal values for the orbit. These nominal values were Brouwer's mean elements pulled from the satellite's TLE. These scaled up CUT elements were then propagated to each time instant to find the coefficients corresponding to each basis function. These coefficients along with the derivative of the basis functions were used to create the direct mapping from the measurement space to the state space. From there the non-linear least square regression was performed to estimate the initial orbit. Measurement data was created by propagating the nominal orbit and adding Gaussian noise of the order 10^{-2} and taking position only in the ECI frame. These results were validated using a Monte-Carlo simulation to support the coefficients ability to accurately estimate the dynamics, and the estimated orbits were compared to the truth value of the nominal orbit propagation using SGP4. The regression was able to converge, however additional changes to the NLS process including the implementation of the Levenberg-Marquardt algorithm could allow faster convergence to within a given tolerance.

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REFERENCES

[1] Thomas Sean Kelso, David A Vallado, Joseph Chan, Bjorn Buckwalter, et al. Improved conjunction analysis via collaborative space situational awareness. In *9th Advanced Maui Optical and Space Surveillance Technologies Conference, Maui, HI*, 2008.

[2] Bob Schutz, Byron Tapley, and George H Born. *Statistical orbit determination*. Elsevier, 2004.

[3] David Vallado and Paul Crawford. Sgp4 orbit determination. In *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, page 6770, 2008.

[4] MultiMedia LLC. Rapid initial orbit determination, 2021.

[5] David A Vallado, Paul Crawford, Richard Hujsak, and TS Kelso. Revisiting spacetrack report# 3. *AIAA*, 6753(2006):446, 2006.

[6] David A Vallado and Paul J Cefola. Two-line element sets-practice and use. In *63rd International Astronautical Congress, Naples, Italy*, pages 1–14, 2012.

[7] J. B. Roberts and P. D. Spanos. *Random vibration and statistical linearization*. Courier Corporation, 2003.

[8] Dirk Brouwer. Solution of the problem of artificial satellite theory without drag. Technical report, YALE UNIV NEW HAVEN CT NEW HAVEN United States, 1959.

[9] Yoshihide Kozai. The motion of a close earth satellite. *The Astronomical Journal*, 64:367, 1959.

[10] S. Shuster. A Survey and Performance Analysis of Orbit Propagators for LEO, GEO, and Highly Elliptical Orbitsn. Master's thesis, Utah State University, 2011.

[11] Manoranjan Majji, John L Junkins, and James D Turner. A high order method for estimation of dynamic systems. *The Journal of the Astronautical Sciences*, 56(3):401–440, 2008.

[12] Ryan S Park and Daniel J Scheeres. Nonlinear mapping of gaussian statistics: theory and applications to spacecraft trajectory design. *Journal of guidance, Control, and Dynamics*, 29(6):1367–1375, 2006.

[13] K Fujimoto, DJ Scheeres, and KT Alfriend. Analytical nonlinear propagation of uncertainty in the two-body problem. *Journal of Guidance, Control, and Dynamics*, 35(2):497–509, 2012.

[14] N. Adurthi, P. Singla, and T. Singh. Conjugate unscented transformation: Applications to estimation and control. *Journal of Dynamic Systems, Measurement, and Control*, 2017.

[15] A. H. Stroud and D. Secrest. *Gaussian Quadrature Formulas*. Englewood Cliffs, NJ: Prentice Hall, 1966.

[16] N. Adurthi, P. Singla, and T. Singh. The conjugate unscented transform-an approach to evaluate multi-dimensional expectation integrals. Proceedings of the American Control Conference, June 2012.

[17] N. Adurthi, P. Singla, and T. Singh. Conjugate unscented transform rules for uniform probability density functions. Proceedings of the American Control Conference, June 2013.

[18] N. Adurthi, P. Singla, and T. Singh. Conjugate unscented transform and its application to filtering and stochastic integral calculation. AIAA Guidance, Navigation, and Control Conference, august 2012.

[19] N. Adurthi and P. Singla. A conjugate unscented transformation based approach for accurate conjunction analysis. *AIAA Journal of Guidance, Control and Dynamics*, 38(9):1642–1658, Sep. 2015.

[20] Thomas Gerstner and Michael Griebel. Numerical integration using sparse grids. *Numerical Algorithms*, 18:209–232, 1998. 10.1023/A:1019129717644.

[21] S.J. Julier, J.K. Uhlmann, and H.F. Durrant-Whyte. A new method for the nonlinear transformation of means and covariances in filters and estimators. *IEEE Transactions on Automatic Control*, AC-45(3):477–482, March 2000.

[22] N. Adurthi. The conjugate unscented transform - a method to evaluate multidimensional expectation integrals. Master's thesis, University at Buffalo, 2013.